

Packing large trees of consecutive orders

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Abstract

A conjecture by Bollobás from 1995 (which is a weakening of the famous Tree Packing Conjecture by Gyárfás from 1976) states that any set of k trees $T_n, T_{n-1}, \dots, T_{n-k+1}$, such that T_{n-i} has $n-i$ vertices, pack into K_n , provided n is sufficiently large. We confirm Bollobás conjecture for trees $T_n, T_{n-1}, \dots, T_{n-k+1}$, such that T_{n-i} has $k-1-i$ leaves or a pending path of order $k-1-i$. As a consequence we obtain that the conjecture is true for $k \leq 5$.

1 Introduction

A set of (simple) graphs G_1, G_2, \dots, G_k are said to *pack into a complete graph* K_n (in short pack) if G_1, G_2, \dots, G_k can be found as pairwise edge-disjoint subgraphs in K_n . Many classical problems in Graph Theory can be stated as packing problems. In particular, H is a subgraph of G if and only if H and the complement of G pack.

A famous tree packing conjecture (TPC) posed by Gyárfás [7] states that any set of n trees T_n, T_{n-1}, \dots, T_1 such that T_i has i vertices pack into K_n . A number of partial results concerning the TPC are known. In particular Gyárfás and Lehel [7] showed that the TPC is true if each tree is either a path or a star. An elegant proof of this result was given by Zaks and Liu [11]. Recently, Böttcher et al. [4] proved an asymptotic version of the TPC for trees with bounded maximum degree (see also [9] for generalizations on other families of graphs). In [6] Bollobás suggested the following weakening of TPC

Conjecture 1 *For every $k \geq 1$ there is an $n_0(k)$ such that if $n > n_0(k)$, then any set of k trees $T_n, T_{n-1}, \dots, T_{n-k+1}$ such that T_{n-j} has $n-j$ vertices pack into K_n .*

Bourgeois, Hobbs and Kasiraj [3] showed that any three trees T_n, T_{n-1}, T_{n-2} pack into K_n . Recently, Balogh and Palmer [2] proved that any set of $k = \frac{1}{10}n^{1/4}$ trees T_n, \dots, T_{n-k+1} such that no tree is a star and T_{n-j} has $n-j$ vertices pack into K_n . In this paper we confirm the conjecture for new sets of trees.

We say that a tree T has a *pending path* of order t if there exists $e \in E(T)$ such that one component of $T - e$ is a path P of order t and $d_T(v) \leq 2$ for every $v \in V(P)$.

Theorem 2 *Let k be a positive integer and let $n_0(k)$ be a sufficiently large constant depending only on k . If $n > n_0(k)$, then any set of k trees $T_n, T_{n-1}, \dots, T_{n-k+1}$, such that T_{n-j} has $n-j$ vertices, and T_{n-j} has $k-1-j$ leaves or a pending path of order $k-1-j$, pack into K_n .*

As an immediate consequence we obtain the following corollary.

Corollary 3 *Let $k \leq 5$ be a positive integer and let $n_0(k)$ be a sufficiently large constant depending only on k . If $n > n_0(k)$, then any set of k trees $T_n, T_{n-1}, \dots, T_{n-k+1}$, such that T_{n-j} has $n-j$ vertices pack into K_n .*

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The proofs of preparatory Lemmas 6 and 9 are inspired by Alon and Yuster approach [1], but are much more involved.

In what follows we fix an integer $k \geq 1$ and assume that $n \geq n_0(k)$, where $n_0(k)$ is a sufficiently large constant depending only on k .

2 Notation

The notation is standard. In particular $d_G(v)$ (abbreviated to $d(v)$ if no confusion arises) denotes the degree of a vertex v in G , $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of G , respectively. Furthermore, $N_G(v)$ denotes the set of neighbors of v and, for a subset of vertices $W \subseteq V(G)$,

$$N_G(W) = \bigcup_{w \in W} N_G(w) \setminus W$$

and

$$N_G[W] = N_G(W) \cup W.$$

Let G be a graph and W any set with $|V(G)| \leq |W|$. Given an injection $f : V(G) \rightarrow W$, let $f(G)$ denote the graph defined as follows

$$f(G) = (W, \{f(u)f(v) : uv \in E(G)\}).$$

For two graphs G and H let $G \oplus H$ denote the graph defined by

$$G \oplus H = (V(G) \cup V(H), E(G) \cup E(H))$$

(note that $V(G)$ and $V(H)$ do not need to be disjoint).

A packing of k graphs G_1, \dots, G_k with $|V(G_j)| \leq n$, $j = 1, \dots, k$, into a complete graph K_n is a set of k injections $f_j : V(G_j) \rightarrow V(K_n)$, $j = 1, \dots, k$ such that

$$\text{if } i \neq j \text{ then } E(f_i(G_i)) \cap E(f_j(G_j)) = \emptyset.$$

For two graphs G and H with $|V(G)| \leq |V(H)|$, we sometimes use an alternative definition. Namely, we call an injection $f : V(G) \rightarrow V(H)$ a packing of G and H , if $E(f(G)) \cap E(H) = \emptyset$.

3 Preliminaries

We write $\text{Bin}(p, n)$ for the binomial distribution with n trials and success probability p . Let $X \in \text{Bin}(n, p)$. We will use the following two versions of the Chernoff bound which follows from formulas (2.5) and (2.6) from [8] by taking $t = 2\mu - np$ and $t = np - \mu/2$, respectively.

If $\mu \geq E[X] = np$ then

$$\Pr[X \geq 2\mu] \leq \exp(-\mu/3) \tag{1}$$

On the other hand, if $\mu \leq E[X] = np$ then

$$\Pr[X \leq \mu/2] \leq \exp(-\mu/8). \tag{2}$$

Proposition 4 *Let G be a graph with n vertices and at most m edges. Let $V(G) = \{v_1, \dots, v_n\}$ with $d(v_1) \geq d(v_2) \geq \dots \geq d(v_n)$. Then*

$$d(v_i) \leq \frac{2m}{i}.$$

Proof. The proposition is true because

$$2m \geq \sum_{j=1}^n d(v_j) \geq \sum_{j=1}^i d(v_j) \geq id(v_i).$$

□

The following technical lemma is the main tool in the proof. A version of it appeared in [1].

Lemma 5 *Let G be a graph with n vertices and at most m edges. Let $V(G) = \{v_1, \dots, v_n\}$ with $d(v_1) \geq d(v_2) \geq \dots \geq d(v_n)$. Let A_i , $i = 1, \dots, n$, be any subsets of $V(G)$ with the additional requirement that if $u \in A_i$ then $d(u) < a$. For $i = 1, \dots, n$ let B_i be a random subset of A_i where each vertex of A_i is independently selected to B_i with probability $p < 1/a$. Let*

$$C_i = \left(\bigcup_{j=1}^{i-1} B_j \right) \cap N(v_i),$$

$$D_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} N[B_j] \right).$$

Then

1. $Pr[|C_i| \geq 4mp] \leq \exp(-2mp/3)$ for $i = 1, \dots, n$
2. $Pr\left[|D_i| \leq \frac{p|A_i|}{2e}\right] \leq \exp\left(-\frac{p|A_i|}{8e}\right)$ for $i = 1, \dots, \lfloor 1/(ap) \rfloor$.

Proof. Fix some vertex $v_i \in V(G)$.

Consider the first part of the lemma. If $d(v_i) \leq 2mp$ then the probability is zero because $|C_i| \leq |N(v_i)| = d(v_i)$. So we may assume that $d(v_i) > 2mp$. For $u \in N(v_i)$ the probability that $u \in B_j$ is at most p (it is either p if $u \in A_j$ or 0 if $u \notin A_j$.) Thus $Pr[u \in C_i] \leq (i-1)p$. By Proposition 4, $i \leq 2m/d(v_i)$. Hence,

$$Pr[u \in C_i] \leq \frac{2mp}{d(v_i)}.$$

Observe that $|C_i|$ is a sum of $d(v_i)$ independent indicator random variables each of which has success probability at most $\frac{2mp}{d(v_i)}$. Thus, the expectation of $|C_i|$ is at most $2mp$. Therefore, by (1), the probability of $|C_i|$ being larger than $4mp$ satisfies

$$Pr[|C_i| \geq 4mp] \leq \exp(-2mp/3).$$

Consider now the second part of the lemma. Observe that for $u \in A_i$, the probability that $u \in B_i$ is p . On the other hand, for any j , the probability that $u \notin N[B_j]$ is at least $1 - ap$. Indeed, $u \in N[B_j]$ if and only if $u \in B_j$ or one of its neighbors belongs to B_j . Since $u \in A_i$, it has at most $a-1$ neighbors. Hence, the probability that $u \in N[B_j]$ is at most ap . Therefore, as long as $i \leq 1/(ap)$,

$$Pr[u \in D_i] \geq p(1 - ap)^{i-1} \geq \frac{p}{e}.$$

Observe that $|D_i|$ is a sum of $|A_i|$ independent indicator random variables, each having success probability at least $\frac{p}{e}$. Therefore the expectation of $|D_i|$ is at least $\frac{p|A_i|}{e}$. By (2), the probability that $|D_i|$ falls below $\frac{p|A_i|}{2e}$ satisfies

$$Pr\left[|D_i| \leq \frac{p|A_i|}{2e}\right] \leq \exp\left(-\frac{p|A_i|}{8e}\right).$$

□

4 Packing trees with small maximum degree.

Lemma 6 *Let G be a graph of order n with $|E(G)| \leq kn$ and $\Delta(G) < 2n/3 + o(n)$. Let T be a tree with $|V(T)| \leq n$ and $\Delta(T) < 60(2k+1)n^{3/4}$. Let $I \subset V(G)$ with $|I| \leq k$ and such that if $v \in I$ then $d_G(v) \leq 2k$. Furthermore, let $I' \subset V(T)$ with $|I'| = |I|$ and such that if $v' \in I'$ then $d_T(v') \leq 2$. Suppose, there is a packing $h' : I' \rightarrow I$ of $T[I']$ and $G[I]$. Then, there is a packing $f' : V(T) \rightarrow V(G)$ of T and G such that*

1. $\Delta(f'(T) \oplus G) \leq 2n/3 + o(n)$,
2. $f'(v') = h'(v')$ for every $v' \in I'$.

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ where $d_G(v_i) \geq d_G(v_{i+1})$. Let G' be a forest that arises from T by adding $n - |V(T)|$ isolated vertices. Let $V(G') = \{v'_1, \dots, v'_n\}$ where $d_{G'}(v'_i) \geq d_{G'}(v'_{i+1})$. For convenience, we will construct a packing $f : V(G) \rightarrow V(G')$ such that $f(h'(v')) = v'$ for every $v' \in V(T)$. Thus for f' we may take f^{-1} restricted to $V(T)$.

Let $A_i \subset V(G) \setminus N_G[v_i]$ with the assumption that if $u \in A_i$ then $d_G(u) < 26k$.

Claim 7 $|A_i| \geq \frac{n}{4}$

Proof. By the assumption on $\Delta(G)$, each vertex of G has at least $n/3 - o(n)$ non-neighbors. Suppose that α vertices of G have degree greater than or equal to $26k$. Thus

$$2kn \geq 2|E(G)| = \sum_{i=1}^n d(v_i) \geq \alpha \cdot 26k,$$

and so $\alpha \leq \frac{n}{13}$. Therefore

$$|A_i| \geq n/3 - o(n) - n/13 \geq n/4.$$

□

For $i = 1, \dots, n$ let B_i be a random subset of A_i where each vertex of A_i is independently selected to B_i with probability

$$p = \frac{n^{-3/4}}{540 \cdot 26k^2(2k+1)} \quad (3)$$

Let

$$C_i = \left(\bigcup_{j=1}^{i-1} B_j \right) \cap N_G(v_i),$$

$$D_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} N_G[B_j] \right).$$

Claim 8 *The following hold with positive probability:*

1. $|C_i| \leq \frac{n^{1/4}}{240(2k+1)}$ for $i = 1, \dots, n$
2. $|D_i| \geq k(2k+1) + 3$ for $i = 1, \dots, \lfloor 540k(2k+1)n^{3/4} \rfloor$.

Proof. Recall that $|E(G)| \leq kn$. Thus, by Lemma 5, the probability that $|C_i| > \frac{n^{1/4}}{240(2k+1)}$ ($> 4|E(G)|p$), is exponentially small in $n^{1/4}$. Hence, for sufficiently large n

$$\Pr \left[|C_i| > \frac{n^{1/4}}{240(2k+1)} \right] < \frac{1}{2n}.$$

Therefore, by the union bound, the first statement holds with probability greater than $1/2$. Furthermore, by Claim 7,

$$k(2k+1) + 3 < \frac{p|A_i|}{2e}.$$

Hence, by Lemma 5 (with $a = 26k$), for each $i \leq \lfloor 540k(2k+1)n^{3/4} \rfloor$ the probability that $|D_i| < k(2k+1) + 3$ is exponentially small in $n^{1/4}$, as well. Hence, for sufficiently large n

$$\Pr[|D_i| < k(2k+1) + 3] < \frac{1}{2n}.$$

Therefore, by the union bound, the second statement holds with probability greater than $1/2$, and so both statements hold with positive probability. \square

Therefore, we may fix sets B_1, \dots, B_n satisfying all the conditions of Claim 8 with respect to the cardinalities of the sets C_i and D_i . We construct a packing $f : V(G) \rightarrow V(G')$ in three stages. At each point of the construction, some vertices of $V(G)$ are *matched* to some vertices of $V(G')$, while the other vertices of $V(G)$ and $V(G')$ are yet *unmatched*. Initially, all vertices are unmatched. We always maintain the packing property, that is for any $u, v \in V(G)$ if $uv \in E(G)$ then $f(u)f(v) \notin E(G')$. The additional requirement that $\Delta(f(G) \oplus G') \leq 2n/3 + o(n)$ is preserved due to the assumption on $\Delta(T) = \Delta(G')$.

After a forced Stage 1, in Stage 2 we match certain number of vertices of G that have the largest degrees. After this stage, by the assumption on $\Delta(G')$, neither G nor G' has unmatched vertices of high degree (vertices of high degree are the main obstacle in packing). This fact enables us to complete the packing in Stages 3 and 4.

Stage 1 In Stage 1 we set $f(h'(v')) = v'$ for each $v' \in I'$. Clearly, the packing property is preserved.

Stage 2 Let x be the largest integer such that $d_{G'}(v_x) \geq \frac{n^{1/4}}{270(2k+1)}$. Thus, by Proposition 4,

$$x \leq 540k(2k+1)n^{3/4} \quad (4)$$

This stage is done repeatedly for $i = 1, \dots, x$ and throughout it we maintain the following two invariants

1. At iteration i we match v_i with some vertex $f(v_i)$ of G' such that $d_{G'}(f(v_i)) \leq 3$.
2. Furthermore, we also make sure that all neighbors of $f(v_i)$ in G' are matched to vertices of $\bigcup_{j=1}^i B_j \cup I$.

To see that this is possible, consider the i 'th iteration of Stage 1 where v_i is some yet unmatched vertex of G . Let Q' be the set of all yet unmatched vertices of G' having degree less than or equal to 3. Note that, by Proposition 4, the number of vertices of degree less than or equal to 3 in G' is at least $n/2$. Hence,

$$|Q'| \geq n/2 - 4(i-1) - k \geq n/2 - 4x - k \geq n/3.$$

Let X be the set of already matched neighbors of v_i and let $Y' = \bigcup_{u \in X} N_{G'}(f(u))$. Thus, the valid choice for $f(v_i)$ would be a vertex of $Q' \setminus Y'$. To see that such a choice is possible, it is enough to show that $|Q'| > |Y'|$. Let $X = X_1 \cup X_2 \cup X_3$ with $X_1 \subseteq I$, $X_2 \subseteq \{v_1, \dots, v_{i-1}\}$ and $X_3 \subseteq B_1 \cup \dots \cup B_{i-1}$. Hence $|X_1| \leq k$, $|X_2| \leq x$ and $|X_3| = |C_i| \leq \frac{n^{1/4}}{240(2k+1)}$. Thus, by the first invariant of Stage 2, and by (4), Claim 8 and the assumptions on I' ,

$$|Q'| - |Y'| \geq n/3 - 2|X_1| - 3|X_2| - \Delta(G')|X_3| \geq n/3 - 2k - 3x - 60(2k+1)n^{3/4} \frac{n^{1/4}}{240(2k+1)} > 0.$$

In order to maintain the second invariant it remains to match the yet unmatched neighbors of $f(v_i)$ with vertices from B_i . Let R' be the set of neighbors of $f(v_i)$ in G' that are still unmatched.

Recall that $|R'| \leq 3$. We have to match vertices of R' with some vertices of B_i . Since $D_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} N_G[B_j]\right)$, a valid choice of such vertices is by taking an $|R'|$ -subset of $D_i \setminus N_G[I]$. By Claim 8 and by (4), $|D_i| \geq k(2k+1) + 3$ for $i = 1, \dots, x$. Furthermore, since each $v \in D_i$ satisfies $d_G(v) < 26k \leq d_G(v_x)$, $D_i \cap \{v_1, \dots, v_{i-1}\} = \emptyset$. Thus, the vertices from $D_i \setminus N_G[I]$ are still unmatched. Since $|N_G[I]| \leq k(2k+1)$ (by the assumptions on I), $|D_i \setminus N_G[I]| \geq 3$. Therefore, such a choice is possible.

Stage 3 Let M_2 and M'_2 be the set of matched vertices of G and G' after Stage 2, respectively. Clearly $|M_2| = |M'_2| \leq 4x + k < n/9$. Hence $G' - M'_2$ has an independent set J' with $|J'| \geq 4n/9$. Let $K' = V(G') \setminus (M'_2 \cup J')$. In Stage 3 we match vertices of K' one by one, with arbitrary yet unmatched vertices of G . Suppose that $v' \in K'$ is still unmatched. Let Q be the set of all yet unmatched vertices of G . Clearly, $|Q| \geq |J'| \geq 4n/9$. Let X' be the set of already matched neighbors of v' . Hence, $|X'| \leq \Delta(G') \leq 60(2k+1)n^{3/4}$. Let $Y = \bigcup_{u' \in X'} N_G(f^{-1}(u'))$. Thus, the valid choice for $f^{-1}(v')$ would be a vertex of $Q \setminus Y$. By the second invariant of Stage 2, $X' \cap \{v_1, \dots, v_x\} = \emptyset$. Hence, by the definition of x ,

$$|Y| \leq |X'| \cdot \frac{n^{1/4}}{270(2k+1)} \leq 60(2k+1)n^{3/4} \cdot \frac{n^{1/4}}{270(2k+1)} \leq 2n/9.$$

Therefore, $|Q \setminus Y| > 0$, and so an appropriate choice for $f^{-1}(v')$ is possible.

Stage 4 Let M_3 and M'_3 be the sets of matched vertices of G and G' after Stage 3, respectively. In order to complete a packing of G and G' , it remains to match the vertices of $V(G) \setminus M_3$ with the vertices of J' . Consider a bipartite graph B whose sides are $V(G) \setminus M_3$ and J' . For two vertices $u \in V(G) \setminus M_3$ and $v' \in J'$, we place an edge $uv' \in E(B)$ if and only if it is possible to match u with v' (by this we mean that mapping u to v' will not violate the packing property). Thus u is not allowed to be matched to at most $d_G(u)\Delta(G')$ vertices of J' . Thus

$$d_B(u) \geq |J'| - \frac{n^{1/4}}{270(2k+1)} 60(2k+1)n^{3/4} \geq |J'|/2.$$

On the other hand, since there is no edge from v' to v_i with $i \leq x$ (by the second invariant of Stage 2), v' is not allowed to be matched to at most $\Delta(G') \frac{n^{1/4}}{270(2k+1)}$ vertices of $V(G) \setminus M_3$. Hence, analogously

$$d_B(v') \geq |J'|/2.$$

Therefore, by Hall's Theorem there is a matching of $V(G) \setminus M_2$ in B , and so a packing of G and G' . \square

5 Packing trees with large maximum degree

Lemma 9 *Let G be a graph of order n with $|E(G)| \leq kn$, $\delta(G) = 0$ and $\Delta(G) < 2n/3 + o(n)$. Let T be a tree with $|V(T)| \leq n$ and $\Delta(T) \geq 60(2k+1)n^{3/4}$. Let $I \subset V(G)$ with $|I| \leq k$ and such that if $v \in I$ then $d_G(v) \leq 2k$. Furthermore, let $I' \subset V(T)$ with $|I'| = |I|$ and such that if $v' \in I'$ then $d_T(v') \leq 2$. Suppose, there is a packing $h' : I' \rightarrow I$ of $T[I']$ and $G[I]$. Then, there is a packing $f' : V(T) \rightarrow V(G)$ of T and G such that*

1. $\Delta(f'(T) \oplus G) \leq 2n/3 + o(n)$,
2. $f'(v') = h'(v')$ for every $v' \in I'$.

Proof. In the proof we will follow the ideas from the previous section. However, the key difference is that now both G and G' may have vertices of high degrees. Because of this obstacle, a packing has two more stages at the beginning. After a preparatory Stage 1, in Stage 2 we match the vertices of G that have high degrees with vertices of G' that have small degrees. Then in Stage 3, we match the vertices of G' having high degree. This stage is very similar to Stage 2 from the

previous section, but with the change of the role of G and G' . Finally, we complete the packing in Stages 4 and 5, which are analogous to Stages 3 and 4 from the previous section.

Let $V(G) = V = \{v_1, \dots, v_n\}$ where $d_G(v_i) \geq d_G(v_{i+1})$. Let G' be a forest that arises from T by adding $n - |V(T)|$ isolated vertices. Let $V(G') = V' = \{v'_1, \dots, v'_n\}$ where $d_{G'}(v'_i) \geq d_{G'}(v'_{i+1})$. For convenience, we will construct a packing $f : V \rightarrow V'$ such that $f(h'(v')) = v'$ for every $v' \in V(T)$. Thus for f' we may take f^{-1} restricted to $V(T)$.

Let $A_i \subset V(G) \setminus N_G[v_i]$ with the assumption that if $u \in A_i$ then $d_G(u) < 26k$. The sets A_i are defined in the same way as in the previous section. Thus,

$$|A_i| \geq \frac{n}{4}. \quad (5)$$

Let

$$q = \frac{n^{1/4}}{59(2k+1)}. \quad (6)$$

Let $P' \subseteq N_{G'}(v'_1)$ be the set of neighbors of v'_1 such that each vertex in P' has degree at most q in G' , and every neighbor different from v'_1 of every vertex from P' has degree at most q in G' .

Claim 10 $|P'| > (2k+1)n^{3/4}$.

Proof. Note that every vertex $v' \in N_{G'}(v'_1) \setminus P'$ has the property that $d_{G'}(v') > q$ or v' has a neighbor $w' \neq v'_1$ such that $d_{G'}(w') > q$. Therefore,

$$n = |V(G')| > (\Delta(G') - |P'|)q \geq (60(2k+1)n^{3/4} - |P'|)\frac{n^{1/4}}{59(2k+1)},$$

and the statement follows. \square

We construct a packing $f : V(G) \rightarrow V(G')$ in five stages. At each point of the construction, some vertices of $V(G)$ are *matched* to some vertices of $V(G')$, while the other vertices of $V(G)$ and $V(G')$ are yet unmatched. Initially, all vertices are unmatched. We always maintain the packing property, that is for any $u, v \in V(G)$ if $uv \in E(G)$ then $f(u)f(v) \notin E(G')$. Furthermore, we preserve that $\Delta(f(G) \oplus G') \leq 2n/3 + o(n)$.

Stage 1. In Stage 1 we set $f(h'(v')) = v'$ for each $v' \in I'$. Furthermore we match an isolated vertex of G with v'_1 , i.e. $f(v_n) = v'_1$.

Stage 2. Let z be the largest integer such that $d_G(v_z) \geq n^{1/4}$. Since $|E(G)| \leq kn$, by Proposition 4

$$z \leq 2kn^{3/4}. \quad (7)$$

This stage is done repeatedly for $i = 1, \dots, z$ and throughout it we maintain the following invariants:

1. At iteration i we match v_i with some vertex $f(v_i)$ of G' such that $f(v_i) \in P' \setminus N_{G'}[I']$.
2. Furthermore, we also make sure that all neighbors of $f(v_i)$ in G' , except v'_1 , are matched to vertices of $A_i \setminus N_G[I]$.

Note that because G' is a forest and since $P' \subseteq N_{G'}(v'_1)$, there are no edges between $N_{G'}[f(v_i)]$ and $N_{G'}[f(v_j)]$ for $i \neq j$. What is more, each $N_{G'}(f(v_j))$ is an independent set in G' . Since there are no edges (in G) between v_i and A_i , the only edges that may spoil the packing property have one endpoint in I or I' . However, by the first invariant there are no edges between I' and $\bigcup_{j=1}^i f(v_j)$, and, by the second invariant, there are no edges between I and $\bigcup_{j=1}^i N_G(v_i) \setminus \{v_1, \dots, v_i\}$. Therefore, such a mapping, if possible, do maintain the packing property. What is more, by (6) and by the definition of P' , the vertices of G having large degrees are matched with vertices of T having small degrees. Subsequently, by the definition of z , the vertices of T having large degrees will be matched with vertices of G having small degrees. Hence, the additional requirement that $\Delta(f(G) \oplus G') \leq 2n/3 + o(n)$ is preserved.

To see that this mapping is indeed possible, consider the i 'th iteration of Stage 2, where v_i is a vertex of G with $d_G(v_i) \geq n^{1/4} \geq 26k$. In particular $v_i \notin \bigcup_{j=1}^{i-1} A_j \cup I$, so v_i is yet unmatched. Note that

$$|P' \setminus N_{G'}[I']| \geq (2k+1)n^{3/4} - 3k \geq z$$

and before iteration i , the number of already matched vertices of $P' \setminus N_{G'}[I']$ was equal to $i-1 < z$. Thus, there is at least one unmatched vertex in $P' \setminus N_{G'}[I']$, say u' , and we may set $f(v_i) = u'$ which preserves the first invariant.

Furthermore, before iteration i the overall number of matched vertices is at most

$$|I| + 1 + (i-1)q < k + 1 + zq \leq k + n/59. \quad (8)$$

Let $R' = N_{G'}(f(v_i)) \setminus \{v_1'\}$. Note that all vertices from R' are still unmatched. Thus, in order to maintain the second invariant, it suffices to match vertices of R' with some vertices of $A_i \setminus N_G[I]$. Observe that by the choice of P' , $|R'| \leq q-1$. Let Q be the set of yet unmatched vertices of $A_i \setminus N_G[I]$. By (5), (8), and since $|N_G[I]| \leq k(2k+1)$,

$$|Q| \geq n/4 - k(2k+1) - (k + n/59) > q-1.$$

Hence, this is possible.

Before we describe Stage 3, we need some preparations. Let M_2 be the set of all vertices of G that were matched in Stage 1 or 2. Similarly, let M_2' be the set of all vertices of G' that were matched in Stage 1 or 2. Recall that

$$|M_2| = |M_2'| \leq k + 1 + zq < k + n/59. \quad (9)$$

Let $H = G[V \setminus M_2]$ be a subgraph of G induced by yet unmatched vertices. Similarly let $H' = G'[V' \setminus M_2']$. Note that since G' is acyclic and by the construction of Stages 1 and 2,

$$d_{G'}(v') \leq d_{H'}(v') + k + 1 \text{ for each } v' \in V' \setminus M_2'. \quad (10)$$

Let $V(H') = \{w'_1, \dots, w'_r\}$ with $d_{H'}(w'_1) \geq d_{H'}(w'_2) \geq \dots \geq d_{H'}(w'_r)$. By (9),

$$r \geq n - (k + n/59) > 3n/4. \quad (11)$$

Let y be the largest integer such that $d_{H'}(w'_y) \geq 360\sqrt{n}$. Then, by Proposition 4,

$$y \leq \frac{2n}{360\sqrt{n}} = \frac{\sqrt{n}}{180}. \quad (12)$$

For each $i = 1, \dots, r$ we define a set $A'_i \subseteq V(H') \setminus N_{H'}[w'_i]$ to be a largest independent set of vertices but with the additional requirement that each $w' \in A'_i$ has $d_{H'}(w') < 180$.

Claim 11 $|A'_i| \geq n/10$, $i = 1, \dots, r$.

Proof. Note that each w'_i has at least

$$r - d_{H'}(w'_i) - 1 \geq r - d_{G'}(w'_i) - 1 \geq r - d_{G'}(v'_2) - k - 1 \geq r - \frac{n}{2} - k - 1 \geq \frac{3}{4}n - \frac{n}{2} - k - 1 = \frac{n}{4} - k - 1$$

non-neighbors. Since H' is a forest, the subgraph of H' induced by all non-neighbors of w'_i has an independent set of cardinality at least $\frac{n/4 - k - 1}{2} > n/9$. Let α be the number of vertices of H' that have degree greater than or equal to 180. Thus

$$2n > \sum_{j=1}^r d_{H'}(w'_j) \geq \alpha \cdot 180,$$

and so $\alpha \leq \frac{n}{90}$. Therefore

$$|A'_i| \geq n/9 - \frac{n}{90} = n/10.$$

□

For $i = 1, \dots, r$ let B'_i be a random subset of A'_i where each vertex of A'_i is independently selected to B'_i with probability $1/\sqrt{n}$. Let

$$C'_i = \left(\bigcup_{j=1}^{i-1} B'_j \right) \cap N_{H'}(w'_i),$$

$$D'_i = B'_i \setminus \left(\bigcup_{j=1}^{i-1} N_{H'}[B'_j] \right).$$

Claim 12 *The following hold with positive probability:*

1. $|C'_i| \leq 4\sqrt{n}$ for $i = 1, \dots, r$
2. $|D'_i| \geq \frac{\sqrt{n}}{20e}$ for $i = 1, \dots, y$.

Proof. Clearly, $|E(H')| < n$. By Lemma 5 (with $m = n$, $p = 1/\sqrt{n}$ and $A_i = A'_i$), the probability that $|C'_i| \geq 4\sqrt{n}$ is exponentially small in \sqrt{n} . Thus, for n sufficiently large

$$\Pr[|C'_i| \geq 4\sqrt{n}] < \frac{1}{2n} \leq \frac{1}{2r}.$$

Furthermore, by Claim (11),

$$\frac{\sqrt{n}}{20e} \leq \frac{p|A'_i|}{2e}.$$

Hence, by the second part of Lemma 5 (with $a = 180$ and the remaining parameters as before) the probability that $|D'_i| \leq \frac{\sqrt{n}}{20e}$ is exponentially small in \sqrt{n} for $i = 1, \dots, \lceil \sqrt{n}/180 \rceil$. Thus, by (12), for $i \leq y \leq \lfloor \sqrt{n}/180 \rfloor$ we have

$$\Pr\left[|D'_i| \leq \frac{\sqrt{n}}{20e}\right] < \frac{1}{2y}.$$

Thus, by the union bound, each part of the lemma holds with probability greater than $1/2$. Hence both hold with positive probability. □

Now we are in the position to describe the next stages of a packing. By Claim 12 we may fix independent sets B'_1, \dots, B'_r satisfying all the conditions of Claim 12 with respect to the cardinalities of the sets C'_i and D'_i . Let $W = \{v_1, \dots, v_z\}$. Recall that

$$\Delta(G - W) < n^{1/4}. \tag{13}$$

Stage 3 This stage is done repeatedly for $i = 1, \dots, y$ and throughout it we maintain the following two invariants

1. At iteration i we match $w'_i \in V(H')$ with some yet unmatched vertex $u = f^{-1}(w'_i)$ of H such that $d_G(u) \leq 4k$.
2. Furthermore, we also make sure that all neighbors of $f^{-1}(w'_i)$ in H are matched to vertices of $\bigcup_{j=1}^i B'_j$.

To see that this is possible, consider the i 'th iteration of Stage 3. Recall that $d_{H'}(w'_i) \geq 360\sqrt{n} \geq 180$. Hence, w'_i does not belong to any B'_j and so it is still unmatched. Let Q be the set of all yet unmatched vertices of G having degree less than or equal to $4k$. Note that, by Proposition 4, the number of vertices of degree less than or equal to $4k$ in G is at least $n/2$. Hence, by (9) and (12)

$$|Q| \geq n/2 - |M_2| - (4k+1)y \geq n/2 - k - n/59 - (4k+1)\sqrt{n}/180 > n/4. \tag{14}$$

Let X' be the set of already matched neighbors in G' of w'_i and let $Y = \bigcup_{x' \in X'} N_G(f^{-1}(x'))$. Thus, the valid choice for $f^{-1}(w'_i)$ would be a vertex of $Q \setminus Y$. We will show that $|Q \setminus Y| > 0$. Let $X' = X'_1 \cup X'_2 \cup X'_3$ such that $X'_1 \subset M'_2$, $X'_2 \subset \{w'_1, \dots, w'_{i-1}\}$ and $X'_3 \subset \bigcup_{j=1}^{i-1} B'_j$. By (10), $|X'_1| \leq k+1$. Moreover if $v' \in X'_1$ then, by the second invariant of Stage 2, $v' \in M'_2 \setminus \{f(v_1), \dots, f(v_z)\}$. Hence, either $f^{-1}(v') \in I$ or $f^{-1}(v')$ belongs to some set A_j , $j \in \{1, \dots, z\}$. Therefore, $d_G(f^{-1}(v')) \leq 26k$. Furthermore, $|X'_2| \leq i-1$ and, by Claim 12, $|X'_3| = |C'_i| \leq 4\sqrt{n}$. Hence, by (13) and by the first invariant of Stage 3,

$$|Y| \leq 26k|X'_1| + 4k|X'_2| + |X'_3| \cdot n^{1/4} < n/4 \quad (15)$$

Therefore, by (14), $|Q \setminus Y| > 0$.

In order to maintain the second invariant we have to match yet unmatched neighbors of $f^{-1}(w'_i)$ with some vertices of B'_i . Let R be the set of the neighbors of $f^{-1}(w'_i)$ in G that are still unmatched. Recall that $|R| \leq 4k$. Since $D'_i = B'_i \setminus \left(\bigcup_{j=1}^{i-1} N_{H'}[B'_j]\right)$, a natural choice of such vertices is by taking an $|R|$ -subset of D'_i . However, unlike in Stage 2 in the previous subsection, this subset cannot be chosen arbitrarily because of the existence of possible edges between vertices from $P'' := I' \cup N_{G'}(P') \setminus \{v'_1\}$ and D'_i . For this reason, we have to match the vertices from R more carefully. We match them, one by one, with some vertices from D'_i in the following way. Suppose that $v \in R$ is yet unmatched. Let D' be the set of yet unmatched vertices of D'_i . Since each $w' \in D'_i$ satisfies $d_{H'}(w') < 180 \leq 360\sqrt{n}$, $D'_i \cap \{w'_1, \dots, w'_{i-1}\} = \emptyset$. Hence,

$$|D'| \geq |D'_i| - |R| \geq \sqrt{n}/(20e) - 4k. \quad (16)$$

Let X_2 be the set of all already matched neighbors of v such that $f(X_2) \subseteq P''$. Let $Y'_2 = \bigcup_{u \in X_2} N_{G'}(f(u))$. Thus, the valid choice for $f(v)$ would be a vertex from $D' \setminus Y'_2$. Recall, that by the definition of z , $|X_2| \leq d_G(v) \leq n^{1/4}$. Furthermore, by the definition of P' and I' , $|N_{G'}(f(u))| \leq q$. Thus, by (6) and (16),

$$|D' \setminus Y'_2| > \sqrt{n}/(20e) - 4k - |X_2|q \geq \sqrt{n}/(20e) - 4k - \sqrt{n}/59 > 0.$$

Thus, an appropriate choice for $f(v)$ is possible.

Stage 4 Let M_3 be the set of matched vertices of G after Stage 3. Similarly, let M'_3 be the set of matched vertices of G' after Stage 3. Note that, by (12) and (9),

$$|M_3| = |M'_3| \leq |M_2| + (4k+1)y \leq k + n/59 + (4k+1)\sqrt{n}/180 < n/4 \quad (17)$$

By (10),

$$\Delta(G' - M'_3) \leq \Delta(H' - M'_3) + k + 1 \leq 360\sqrt{n} + k + 1. \quad (18)$$

Furthermore, $|V(G') \setminus M'_3| > n - n/4 = 3n/4$. Thus $G' - M'_3$ has an independent set J' with $|J'| > 3n/8$. Let $K' = V(G') \setminus (J' \cup M'_3)$. In Stage 4 we match vertices from K' one by one, with arbitrary yet unmatched vertices of G . Suppose that $v' \in K'$ is still unmatched. Let Q be the set of all yet unmatched vertices of G . Clearly, $|Q| \geq |J'| \geq 3n/8$. Let X' be the set of already matched neighbors of v' . By (18), $|X'| \leq 360\sqrt{n} + k + 1$. Let $Y = \bigcup_{x' \in X'} N_G(f^{-1}(x'))$. Thus, the valid choice for $f^{-1}(v')$ would be a vertex of $Q \setminus Y$. By the second invariant of Stage 2, $X' \cap \{v_1, \dots, v_x\} = \emptyset$. Hence, by (13),

$$|Y| \leq |X'| \cdot n^{1/4} < 3n/8 - 1.$$

Hence

$$|Q \setminus Y| \geq 1,$$

and so the choice for $f^{-1}(v')$ is possible.

Stage 5 Let M_4 and M'_4 be the sets of matched vertices of G and G' , respectively, after Stage 4. In order to complete a packing of G and G' it remains to match the yet unmatched

vertices of G with vertices of J' . Consider a bipartite graph B whose sides are $J := V(G) \setminus M_4$ and J' . For two vertices $u \in J$ and $v' \in J'$, we place an edge $uv' \in E(B)$ if and only if it is possible to match u with v' (by this we mean that mapping u to v' will not violate the packing property). Recall that, by (13), $d_G(u) \leq n^{1/4}$. Moreover, by the second invariant of Stage 3, $f(N_G(u)) \subset V(G') \setminus \{w'_1, \dots, w'_y\}$. Thus, by the definition of y and by (10), u is not allowed to be matched to at most $n^{1/4} (360\sqrt{n} + k + 1)$ vertices of J' . Therefore,

$$d_B(u) \geq |J'| - n^{1/4} (360\sqrt{n} + k + 1) > |J'|/2.$$

Similarly, $d_{G'}(v') \leq 360\sqrt{n} + k + 1$. Moreover, $f^{-1}(N_{G'}[v']) \subset V(G) \setminus W$. Thus, by (13),

$$d_B(v') \geq |J'| - n^{1/4} (360\sqrt{n} + k + 1) > |J'|/2.$$

Therefore, by Hall's Theorem there is a perfect matching in B , and so a packing of G and G' . \square

6 Proof of Theorem 2

Recall the theorem of Gyárfás and Lehel [7].

Theorem 13 *Let T_1, \dots, T_q be trees of orders $1, \dots, q$, respectively. If each T_j is either a path or a star, then there exists a packing of T_j , $j = 1, \dots, q$, into K_q .*

We will also need the following theorem proved by Brandt [5].

Theorem 14 *For every $0 < \alpha < 1/2$, there exists $n_0 = n_0(\alpha)$ such that if $n > n_0$, $|E(G_1)| \leq \alpha n$ and $|E(G_2)| \leq \frac{1}{3\sqrt{\alpha}} n^{3/2}$, then G_1 and G_2 pack.*

Proof of Theorem 2. We say that T_{n-j} , $j = 0, \dots, k-1$, is of type I if $\Delta(T_{n-j}) < 60(2k+1)n^{3/4}$, of type II if $60(2k+1)n^{3/4} \leq \Delta(T_{n-j}) < 2n/3$, and of type III if $2n/3 \leq \Delta(T_{n-j})$. By the assumption, for each j there exists a set $A_j \subset V(T_{n-j})$ such that either A_j consists of $k-j-1$ leaves or A_j is the vertex set of a pending path of order $k-j-1$. What is more, if T_{n-j} is of type III, then it has a set A_j which consists of $k-j-1$ leaf-neighbors of the vertex of maximum degree. If A_j is the vertex set of a pending path then let $I_j = A_j \cup \{l\}$ where $l \in V(T_{n-j}) \setminus A_j$ is a leaf of T_{n-j} (clearly, such l does exist). Otherwise let $I_j = A_j \cup \{w_j\}$ where $d_{T_{n-j}}(w_j) = \Delta(T_{n-j})$. Let $T_{k-j} = T_{n-j}[I_j]$. Thus $T_{k-j} = P_{k-j-1} \cup K_1$ or T_{k-j} is a subgraph of the star $K_{1, k-j-1}$. In the former case let T_{k-j}^* be a path that arises from T_{k-j} by adding the missing edge between l and a vertex of degree 2 in T_{n-j} . In the latter case, let T_{k-j}^* be the star with the center w_j that arises from T_{k-j} by adding the missing edges incident to w_j . Note that if T_{n-j} is of type III, then T_{k-j} is the star $K_{1, k-j-1}$. In particular for $j \in \{k-1, k-2, k-3\}$, T_{k-j}^* is included to paths if T_{n-j} is of type I, and T_{k-j}^* is included to stars if T_{n-j} is of type II or III. Clearly

$$d_{T_{n-j}}(v) \leq d_{T_{k-j}^*}(v) \text{ for every } v \in I_j \setminus w_j. \quad (19)$$

Let G_0 be a graph with vertex set $V = \{v_1, \dots, v_n\}$ and without edges. Let $K = \{v_{n-k+1}, \dots, v_n\}$. By Theorem 13, there exists a packing $h_j : V(T_{k-j}^*) \rightarrow K$ of T_{k-j}^* , $j = 0, \dots, k-1$.

Let T_{n-s_i} , $i = 1, \dots, s$, are of type III. We say that an edge $uv \in E(T_{k-j}^*)$ is *redundant* in T_{k-j}^* (with respect to h_j , $j = 0, \dots, k-1$), if

$$\left| \{h_j(u), h_j(v)\} \cap \bigcup_{i=1}^s h_{s_i}(w_{s_i}) \right| \leq 1.$$

Otherwise uv is called *essential*.

A more detailed inspection of the short proof of Theorem 13 given by Liu and Zaks [11] (see also [10], p. 67), shows that there exists a packing $h_j : V(T_{k-j}^*) \rightarrow K$, $j = 0, \dots, k-1$, such that

$$\text{if } i > j, \text{ and } T_{k-i}^* \text{ and } T_{k-j}^* \text{ are stars, then } h_i(V(T_{k-i}^*)) \subseteq K \setminus \{h_j(w_j)\}, \quad (20)$$

$$\text{each path has a redundant edge incident to its endvertex.} \quad (21)$$

In particular, if T_{n-j} is of type II then all edges of T_{k-j}^* are redundant. Indeed, (20) implies that $h_j(w_j) \neq h_{s_i}(w_{s_i})$, $i = 1, \dots, s$. On the other hand all the edges of T_{k-j}^* are incident to w_j . Furthermore, by (21), we may assume that if T_{n-j} is of type I then $E(T_{k-j}^*) \setminus E(T_{k-j})$ is a redundant edge. To sum up, we have that

$$T_{k-j} \text{ contains all essential edges of } T_{k-j}^* \quad j = 0, \dots, k-1. \quad (22)$$

Let p, r, s be the numbers of trees of type I, II, and III, respectively. Let P_1, \dots, P_p with $|P_i| < |P_{i+1}|$, $i = 1, \dots, p-1$, denote the trees of type I. Similarly, let R_1, \dots, R_r with $|R_i| < |R_{i+1}|$, and S_1, \dots, S_s , with $|S_i| < |S_{i+1}|$, denote the trees of type II and III, respectively. Let $R_i = T_{n-r_i}$, $i = 1, \dots, r$. We partition K and each I_j into two subsets:

$$\begin{aligned} Y &= \bigcup_{i=1}^r h_{r_i}(w_{r_i}), \\ X &= K \setminus Y, \\ Y_j &= h_j^{-1}(Y), \\ X_j &= h_j^{-1}(X) = I_j \setminus Y_j. \end{aligned}$$

We first pack R_i , $i = 1, \dots, r$, in a special way. We construct injections $f_{r_i} : V(R_i) \rightarrow V$, $i = 1, \dots, r$, having the following properties:

$$\begin{aligned} E(f_{r_i}(R_i)) \cap E(G_{i-1}) &= \emptyset \text{ with } G_i = f_{r_i}(R_i) \oplus G_{i-1}, \\ f_{r_i}(v) &= h_{r_i}(v) \text{ for every } v \in X_{r_i}, \\ f_{r_i}^{-1}(X) &= X_{r_i}, \\ \Delta(G_i) &\leq 2n/3 + o(n). \end{aligned}$$

To see that this is possible, consider the i -th iteration of this constructions. Note that, by (20), $d_{G_{i-1}}(h_{r_i}(w_{r_i})) = 0$. Hence $\delta(G_{i-1}) = 0$. Let $G'_{i-1} = G_{i-1}[V \setminus X \cup h_{r_i}(X_{r_i})]$. Since $h_{r_i}(w_{r_i}) \in Y \subset V \setminus X$, $\delta(G'_{i-1}) = 0$, as well. Let $I' = X_{r_i}$ and $I = h_{r_i}(I') \subseteq X$. Clearly, $d_{R_i}(u) \leq 1$ for each $u \in I'$. Furthermore, $d_{R_i}(h_{r_i}^{-1}(v)) \leq 1$, $t = 1, \dots, i-1$, for every $v \in I$. Hence, $d_{G'_{i-1}}(v) \leq i-1 \leq k$ for every $v \in I$. Therefore, by Lemma 9 with $G = G'_{i-1}$, $T = R_i$ and $h' = h_{r_i}$, an appropriate f_{r_i} does exist. In particular, the third property is preserved because

$$\begin{aligned} f_{r_i}^{-1}(X) &= f_{r_i}^{-1}(h_{r_i}(X_{r_i})) && \text{by the definition of } G'_{i-1} \\ &= f_{r_i}^{-1}(f_{r_i}(X_{r_i})) = X_{r_i} && \text{by the second property.} \end{aligned}$$

Now we pack $P_i := T_{n-p_i}$, $i = 1, \dots, p$. We construct injections $f_{p_i} : V(P_i) \rightarrow V$, $i = 1, \dots, p$, having the following properties:

$$\begin{aligned} E(f_{p_i}(P_i)) \cap E(G_{r+i-1}) &= \emptyset \text{ with } G_{r+i} = f_{p_i}(P_i) \oplus G_{r+i-1}, \\ f_{p_i}(v) &= h_{p_i}(v) \text{ for every } v \in X_{p_i}, \\ f_{p_i}^{-1}(X) &= X_{p_i}, \\ \Delta(G_i) &\leq 2n/3 + o(n). \end{aligned}$$

To see that this is possible, consider the i -th iteration of this constructions. Let $I' = X_{p_i}$ and $I = h_{p_i}(I')$. Clearly, $d_{P_i}(v) \leq 2$ for each $v \in I' \subseteq I_{p_i}$. Furthermore, $d_{P_t}(h_{p_t}^{-1}(v)) \leq 2$, $t = 1, \dots, i-1$,

for every $v \in I \subseteq X$. Hence, $d_{G_{i-1}}(v) \leq r + 2(i-1) \leq 2k$ for every $v \in I$. Therefore, by Lemma 6 with $G = G_{r+i-1}[V \setminus X \cup h_{p_i}(X_{p_i})]$, $T = P_i$ and $h' = h_{p_i}$, an appropriate f_{p_i} does exist.

Finally, we pack $S_i := T_{n-s_i}$, $i = 1, \dots, s$. We distinguish in X and each X_j the following subsets:

$$Z = \bigcup_{i=1}^s h_{s_i}(w_{s_i}),$$

$$Z_j = h_j^{-1}(Z).$$

By (22),

$$|N_{T_{k-j}}(v) \cap Z_j| = |N_{T_{k-j}^*}(v) \cap Z_j| \quad (23)$$

for every $v \in Z_j$, $j = 0, \dots, k-1$.

Let $F_0 = G_{p+r}$. Thus,

$$F_0 = \bigoplus_{j \in [0, k-1] \setminus \{s_1, \dots, s_s\}} f_j(T_{n-j}).$$

Let

$$H_{i-1}^* = \bigoplus_{j \in [0, k-1] \setminus \{s_i, \dots, s_s\}} h_j(T_{k+1-j}^*) \quad i = 1, \dots, s$$

$$H_s^* = \bigoplus_{j \in [0, k-1]} h_j(T_{k+1-j}^*).$$

Let $v_i = h_{s_i}(w_{s_i})$. By (20), $v_i \neq v_j$ for $i \neq j$. We construct injections $f_{s_i} : V(S_i) \rightarrow V$, $i = 1, \dots, s$, having the following properties:

$$E(f_{s_i}(S_i)) \cap E(F_{i-1}) = \emptyset \text{ with } F_i = f_{s_i}(S_i) \oplus F_{i-1},$$

$$f_{s_i}(v) = h_{s_i}(v) \text{ for every } v \in Z_{s_i},$$

$$f_{s_i}^{-1}(Z) = Z_{s_i}.$$

To see that this is possible, consider the i -th iteration of this construction. We set $f_{s_i}(v) = h_{s_i}(v)$ for every $v \in Z_{s_i}$. Note that h_{s_i} sends w_{s_i} and its $\alpha := |Z_{s_i}| - 1$ neighbors to Z . Since $w_{s_i} \in Z_{s_i}$, f_{s_i} sends w_{s_i} and its α neighbors to Z , as well. On the other hand $d_{T_{k-s_i}^*} = k-1-s_i$. Thus, h_{s_i} sends $k-1-s_i-\alpha$ neighbors of w_{s_i} to $K \setminus Z$. Since $h_j : V(T_{k-j}^*) \rightarrow K$, $j = 0, \dots, k-1$, is a packing of T_{k-j}^* ,

$$k-s-\left|N_{H_{i-1}^*}(v_i) \setminus Z\right| \geq k-1-s_i-\alpha. \quad (24)$$

We will show that there is a set $K' \subset (V \setminus Z) \setminus N_{F_{i-1}}(v_i)$ such that $|K'| = n-1-s_i-\alpha$. By (19), (20) and by the construction of f_j

$$\begin{aligned} |N_{F_{i-1}}(v_i)| &= d_{F_{i-1}}(v_i) = \sum_{j \in [0, k-1] \setminus \{s_i, \dots, s_s\}, v_i \in f_j(Z_j)} d_{T_{n-j}}(f_j^{-1}(v_i)) \\ &\leq \sum_{j \in [0, k-1] \setminus \{s_i, \dots, s_s\}, v_i \in h_j(Z_j)} d_{T_{k-j}^*}(h_j^{-1}(v_i)) = d_{H_{i-1}}(v_i) = |N_{H_{i-1}^*}(v_i)|. \end{aligned}$$

Therefore, by (23,24),

$$\begin{aligned} |(V \setminus Z) \setminus N_{F_{i-1}}(v_i)| &\geq n-s-\left|N_{F_{i-1}}(v_i) \setminus Z\right| = n-s-\left|N_{F_{i-1}}(v_i)\right| + \left|N_{F_{i-1}}(v_i) \cap Z\right| \\ &\geq n-s-\left|N_{H_{i-1}^*}(v_i)\right| + \left|N_{H_{i-1}^*}(v_i) \cap Z\right| = n-s-\left|N_{H_{i-1}^*}(v_i) \setminus Z\right| \\ &\geq n-1-s_i-\alpha. \end{aligned}$$

Thus, an appropriate K' does exist.

Let $S'_i = S_i - Z_{s_i}$ and let $G' = F_{i-1}[K']$. Thus, $n' := |V(S'_i)| = n - s_i - \alpha - 1$. In order to complete the construction of f_{s_i} , it is sufficient to pack S'_i and G' . By the definition of trees of type III, $d_{S_i}(w_{s_i}) \geq 2n/3$. Thus, $|E(S'_i)| \leq n/3 \leq \frac{2}{5}n'$. Moreover,

$$|E(G')| \leq kn < \frac{1}{3\sqrt{2/5}}(n')^{3/2}.$$

Thus, by Theorem 14 such a packing does exist. \square

7 Remarks

In the previous version of this paper, we claimed that we proved Bollobás conjecture in full. Unfortunately, the proof contained a mistake. We sincerely appologize the Readers for this misinformation.

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